COMP 232 Mathematics for Computer Science
Fall 2012
Midterm Exam

Name: ______________________ Total Points:

ID: ______________________ ________ / 71

Instructions. This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet. Use pen, not pencil. Do not detach any pages from this exam!

(2$^{\text{pts}}_{\text{ea.}}$) 1. Let the universe of discourse be $\mathbb{Z}^+$, the set of positive integers. For each of the following sentences, indicate whether it is true or false. You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

10 pts

(a) $\forall x\big((x < 0) \vee (x \leq 2x)\big)$

☐ False ☐ True ☐ Don't know!

(b) $\exists x \exists y\big((x + y = 0) \vee (x \cdot y = 0)\big)$

☐ True ☐ False ☐ Don't know!

(c) $\forall x \forall y(x \cdot y \geq x + y)$

☐ False ☐ True ☐ Don't know!

(d) $\exists x \exists y\big((x = 3) \vee (y = 4)\big)$

☐ True ☐ False ☐ Don't know!

(e) $\exists x \forall y \exists z\big((y = x + z) \wedge (z \leq x)\big)$

☐ True ☐ False ☐ Don't know!

10 pts

(6$^{\text{pts}}_{\text{ea.}}$) 2. Here you are to prove propositional equivalences using the laws at the last page of this exam

12 pts

(a) Here is a proof that $p \to (q \to r) \equiv (p \wedge q)$.

Step			Law applied
$p \to (q \to r)$	$\equiv$	$\neg p \vee (q \to r)$	
	$\equiv$	$\neg p \vee (\neg q \vee r)$	
	$\equiv$	$(\neg p \vee \neg q) \vee r$	
	$\equiv$	$\neg(p \wedge q) \vee r$	
	$\equiv$	$(p \wedge q) \to r$	

In the rightmost column above, fill in the law applied for each step (see last page of this booklet for a list of laws)

(b) In the table below, construct a proof of the equivalence

$$(r \vee p) \to (r \vee q) \equiv r \vee (p \to q)$$

similarly to (a).

Step	Law applied
	Implication
	de Morgan
	Associativity
	Commutativity
	Distributivity
	Commutativity
	Excluded middle
	Identity
	Associativity
	Implication

12 pts

(3$^{\text{pts}}_{\text{ea.}}$) 3. We know that $\{\wedge, \neg\}$ forms a functionally complete set of operators, meaning that any other operator can be defined in terms of $\{\wedge, \neg\}$ only, for example

9 pts

$$\begin{aligned} p \vee q &=_{\mathsf{def}} \neg(\neg p \wedge \neg q) \\ p \rightarrow q &=_{\mathsf{def}} \neg(p \wedge \neg q) \end{aligned}$$

The Shaffer stroke $\uparrow$ is a binary operator that has the following truth table:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Show that the Shaffer stroke by itself is functionally complete, by defining in the space below, the following operators by using the Shaffer stroke only:

(a) $\neg p \ =$

(b) $p \wedge q \ =$

(c) $p \vee q \ =$

(6$^{\text{pts}}$) 4. The negation of the statement $\forall x \neg \forall y \exists z (P(x,z) \wedge Q(z,y))$ is

6 pts

- ☐ $\exists x \forall y \exists z (P(x,z) \wedge Q(z,y))$
- ☐ $\forall x \exists y \forall z (\neg P(x,z) \vee \neg Q(z,y))$
- ☐ $\forall x \forall y \exists z (\neg P(x,z) \wedge \neg Q(z,y))$
- ☐ $\forall x \exists y \forall z (\neg P(x,z) \wedge \neg Q(z,y))$
- ☐ $\exists x \exists y \forall z (P(x,z) \vee \neg Q(z,y))$

(6$^{\text{pts}}$) 5. Which of the following statements is the contrapositive of the statement "You win the game if you know the rules but are not overconfident."

6 pts

- ☐ "If you lose the game then you don't know the rules or you are overconfident."
- ☐ "If you don't know the rules or are overconfident then you lose the game."
- ☐ "If you don't know the rules and are overconfident then you win the game."
- ☐ "A sufficient condition that you win the game is that you know the rules or you are not overconfident."
- ☐ "A necessary condition that you know the rules or you are not overconfident is that you win the game."

21 pts

(6pts) 6. To prove $p \wedge (\neg q) \Rightarrow r \vee (\neg s)$ by contradiction, which of the following propositions is the appropriate one to prove.

6 pts

- ☐ $\big((\neg p) \wedge q \wedge s \wedge (\neg r)\big) \Rightarrow False$
- ☐ $\big((\neg p) \wedge q \wedge r \wedge (\neg s)\big) \Rightarrow False$
- ☐ $\big((\neg q) \wedge p \wedge (\neg r) \wedge s\big) \Rightarrow False$
- ☐ $\big((\neg q) \wedge p \wedge s \wedge (\neg r)\big) \Rightarrow False$
- ☐ $\big(p \wedge (\neg q) \wedge (\neg r) \wedge s\big) \Rightarrow False$

(8pts) 7. $|\mathcal{P}\big((A \times B) \cup (B \times A)\big)| = |\mathcal{P}\big((A \times B) \cup (A \times B)\big)|$ <u>if and only if</u>

8 pts

- ☐ $A = \emptyset$ or $B = \emptyset$ or $A \cap B = \emptyset$
- ☐ $A = B$
- ☐ $A = \emptyset$ or $B = \emptyset$
- ☐ $B = \emptyset$ or $A = B$
- ☐ $A = \emptyset$ or $B = \emptyset$ or $A = B$

(2pts ea.) 8. Consider the following proof that there is no least (smallest) positive real number.

6 pts

Proof: Suppose to the contrary that there is a real number x, such that x is positive and ______________ for all positive real numbers y. Consider the number $x/2$. Then ______________ because x is positive. Hence ______________, which is a contradiction.■

In (a), (b), and (c) below, fill in the corresponding blanks in the proof, such that you get a valid proof.

(a) $x \leq y$

(b) $x/2 > 0$

(c) $x \leq x/2$

20 pts

($2^{\text{pts}}_{\text{ea.}}$) 9. The symmetric difference between sets A and B is defined as

8 pts

$$A \oplus B = (A - B) \cup (B - A).$$

For each of the proposed identity involving $\oplus$ below, state whether the identity is true or false. You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

(a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

☐ False ☐ True ☐ Don't know!

(b) $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$.

☐ True ☐ False ☐ Don't know!

(c) $((A \oplus B) \oplus C) \cup (A \cap B \cap C) = A \cup B \cup C$.

☐ False ☐ True ☐ Don't know!

(d) $(A \oplus B) \cup B = A \cup B$

☐ True ☐ False ☐ Don't know!

. — End of Exam — .

8 pts

Basic logical equivalences:

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotency
$\neg(\neg p) \equiv p$	Double negation
$p \vee (\neg p) \equiv T$	Law of excluded middle
$p \wedge (\neg p) \equiv F$	Law of contradiction
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutativity
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributivity
$p \wedge (p \vee q) \equiv p$ $p \vee (p \wedge q) \equiv p$	Absorption laws
$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$	de Morgan's laws

0 pts